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TECHNICAL REPORT

Office of Naval Research Contract No. N00014-86-K0029

ROBUSTNESS OF ROOTFINDING IN QUEUEING ANALYSES

by

Mohan L. Chaudhry
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Report No. GMU/22472/103
August 31, 1988

Department of Operations Research and Applied Statistics
School of Information Technology and Engineering
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Abstract

There has been frequent controversy over the years regarding the use of numerical rootfinding for the solution of queueing problems. It has been said that such problems quite often present fairly typical computational difficulties. However, it turns out that rootfinding in queueing is so well structured that problems do not occur. There are fundamental properties possessed by the well-known queueing models that eliminate classical rootfinding problems. Most importantly, we show that uniqueness of roots is standard within simply determined regions in the complex plane and prove that the $G/E_K/1$ model has unique roots easily found inside the complex unit circle. Extensive computational results are given to support our contentions.

ROBUSTNESS OF ROOTFINDING IN QUEUEING ANALYSES

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1 INTRODUCTION

In all the major textbooks on queueing theory, we see repeated reference to the role of rootfinding in the solution of many of the primary models. A large number of these models can be solved by other numerically-intensive procedures which, however, preserve a much closer link to the underlying stochastic processes. (See especially the work of Neuts, e.g. 1981.) The point is often made that the use of transforms and/or generating functions leads away from probabilistic arguments to the application of analytic function theory. But much of the "anti-transform" position is directed at the typical need to find roots to a transcendental or high-degree polynomial equation in order to finish off a transform solution. It is the common wisdom that such rootfinding is fraught with critical obstacles, such as difficulties raised by multiple roots or sharp slopes in the function, so that some roots may indeed not be found by even the most sophisticated algorithms. However, it turns out instead that rootfinding in queueing is so well structured that these problems do not occur. It is the objective of this work to show that the standard rootfinding problems found in queueing analyses have fundamental properties that always allow computational solution.

It is important to recognize that the efficient finding of roots is essentially a three-phase process. First, we need to determine their multiplicity, that is, whether or not there are any repetitions. Second, roughly, where are the roots? (It is, for example, quite useful to know how many are inside or on the unit circle in the complex plane.) And, finally, we need to develop an effective algorithm and employ it on appropriate hardware to obtain *complete* answers within reasonable amounts of time.

2 SOME KNOWN AND BASIC RESULTS

It has already been shown in a number of queueing models that roots are unique and that they can easily be obtained numerically. (See Chaudhry and Templeton, 1983, Chaudhry, Madill and Brière, 1987, and Brière and Chaudhry, 1987, for example.) We present a short discussion of some of the more familiar ones here (in no particular order) to illustrate what is already known and to lay some groundwork (with proofs streamlined) for the more generic results which follow. Section 3 will provide more complete details on root calculation for the critical models under the most extreme of conditions.

2.1 $M/M^{(K)}/1$

Just about the simplest Markovian queue for which rootfinding becomes germane is the bulk variation of the classic $M/M/1$ in which service is in fixed batches of size K (whether or not the server has to wait for a full batch of size K is irrelevant. The key equation here appears in the denominator of the queue-length generating function (or, equivalently, as the characteristic equation when the stationary system-size probabilities are related in difference-equation form):

$$\rho z^{K+1} - (\rho + 1)z^K + 1 = 0 \qquad (\rho = \lambda/\mu). \tag{1}$$

First, we can show under ergodicity (using Rouché's theorem - see Gross and Harris, 1985, for example) that this equation has one root of 1, K-1 roots inside or on the unit circle, and finally one outside the unit circle. This latter root ultimately determines the values of the stationary probabilities.

Though we do not need to determine the roots inside or on the unit circle to solve the queueing problem, it is useful to look at them a little more closely. It turns out that the roots are unique (as first noted by Bailey, 1954), a property quite typical of the kinds of rootfinding problems arising in stochastic models. This is proven by contradiction, and we start by assuming that $z_i \neq 0$ is a repeater. Since the derivative must then vanish at z_i, it follows that

$$z_i^{K-1}[\rho(K+1)z_i - (\rho+1)K] = 0,$$

thus implying that

$$z_i = \frac{K}{K+1}\frac{\rho+1}{\rho}.$$

By assumption, we see that z_i is a repeated root, and it must be inside or on the unit circle. But

$$| z_i | \leq 1 \text{ iff } \rho/K \geq 1. \tag{2}$$

However, this violates the condition for ergodicity. Hence the roots are distinct. (Of course, we recognize that this is the same construct as that for the $E_K/M/1$ queue.)

In order to find the root outside the unit circle in this case, you can observe that it must be real, since it is unique. Moreover, the function is 0 when $z = 1$ and the derivative of the function is negative in the interval $[1, z^*]$, where $z^* = K(\rho+1)/\rho(K+1)$. Hence we need examine only the real interval greater than z^*.

Newton's method finds successively better approximations for the real root of $F(z) = 0$ via the equation $z_{(n+1)} = z_{(n)} - F(z_{(n)})/F'(z_{(n)})$. A sufficient condition for convergence of this method is that the first two derivatives be positive. This is easily shown for our case when $z > z^*$:

$$F'(z) = z^{K-1}[\rho(K+1)z - (\rho+1)K]$$

has both factors positive; and

$$F''(z) = Kz^{K-2}[\rho(K+1)z - (\rho+1)(K+1)]$$

is positive when $z > (\rho+1)(K-1)/\rho(K+1)$. But this ratio is smaller than z^* and hence the second-order condition for convergence is also satisfied. Thus it appears that Newton's method, perhaps with a starting value of $(\rho+1)/\rho$, is the fastest and easiest way to find the root outside the unit circle. Note that we wish to avoid z^* as a starting point because the derivative vanishes there. (Since the function is a polynomial, with a second derivative easily computed, a variant of Muller's method of approximation by a quadratic might converge even faster.)

A second method for finding the real root outside the unit circle is a fixed-point iteration or successive-substitution approach. The original polynomial expression is rewritten as the $(K+1)st$ root of $[(\rho+1)z^K - 1]/\rho$ and then z is repeatedly substituted into this expression. A sufficient condition for the convergence of this method is that the derivative of this $(K+1)st$ root be between -1 and 1, so that the mapping truly contracts the domain.

This condition can be proven as follows:

$$f'(z) = \frac{1}{K+1}\left[\frac{(\rho+1)z^K - 1}{\rho}\right]^{\frac{-K}{K+1}} \frac{(\rho+1)Kz^{K-1}}{\rho}$$

$$= \frac{K(\rho+1)}{(K+1)\rho} z^{K-1} \left[\frac{(\rho+1)z^K - 1}{\rho}\right]^{\frac{-K}{K+1}}.$$

We then recognize the first factor as z^* defined earlier and restrict our attention only to the interval where $z > z^*$:

$$f'(z) = z^* z^{K-1} \left[\frac{(\rho+1)z^K - 1}{\rho}\right]^{\frac{-K}{K+1}},$$

so that

$$f'(z) < z^K \frac{\rho}{(\rho+1)z^K - 1}$$

$$= \frac{\rho z^K}{(\rho+1)z^K - 1} < 1.$$

Thus, we see that the condition is satisfied and this approach is also guaranteed to converge as long as the starting value exceeds z^*.

2.2 $M^{(K)}/M/1$ or $M/E_K/1$

Here the characteristic equation whose roots we need to find is given by

$$\rho z^{K+1} - (\rho+1)z + 1 = 0. \tag{3}$$

(Of course, we recognize that the state probabilities can be obtained without rootfinding by going directly to the usual recurrence relationship building up from the easily computed p_0, as in Gross and Harris, 1985, Chapter 3.) Equation 3 is known to have a root of 1 and to have the remaining K all outside the unit circle. As in Section 2.1, the lack of multiplicity can be verified here.

First, note that since any repeat root must make the deriviative of the left-hand side of (3) vanish, it follows that z_i would be a repeater if

$$z_i^K = \frac{\rho+1}{\rho(K+1)}.$$

To determine whether or not z_i is a multiple root, we need then to see if it satisfies (3):

$$0 \stackrel{?}{=} \rho z_i^{K+1} - (\rho + 1)z_i + 1 = z_i \frac{\rho + 1}{K + 1} - (\rho + 1)z_i + 1$$

or

$$z_i \stackrel{?}{=} \frac{K+1}{K(\rho+1)}.$$

Thus we need to show that

$$\left[\frac{K+1}{K(\rho+1)}\right]^K = \frac{\rho+1}{\rho(K+1)},$$

or that

$$\left[\frac{K(\rho+1)}{K+1}\right]^{K+1} = K\rho.$$

But these two terms cannot be equal since

$$\left[\frac{K(\rho+1)}{K+1}\right]^{K+1} = \left[1 - \frac{1-K\rho}{K+1}\right]^{K+1}$$
$$> 1 - \frac{(K+1)(1-K\rho)}{K+1} = K\rho.$$

Thus repetition is not possible and the roots are simple.

The actual estimation of the roots of (3) (or more appropriately, (3) transformed by $u = 1/z$ to get the roots inside the unit circle) can be done nicely using the Jenkins-Traub (1970) rootfinding algorithm. This is a cubically convergent application of Newton's method to a rational approximation of the polynomial. Zeros are calculated in roughly increasing order of moduli, which are all less than 1 on the inside of the unit circle. This approach works particularly well because of the isolation of the roots and the fact that deflation once again leaves a polynomial to solve. Purification would be advisable for large values of K to make the technique work more smoothly.

2.3 $M/D/c$

As noted in Gross and Harris (1985), the fundamental root equation for the M/D/c (or equivalently, the $M/D^{(c)}/1$) queue is (with the service rate assumed to be 1)

$$1 - z^c exp[\lambda(1 - z)] = 0, \tag{4}$$

or

$$z^c = exp[-\lambda(1 - z)]. \tag{5}$$

There are c roots inside and on $| z |= 1$, one equal to 1 and the remaining c-1 inside (see Chaudhry and Templeton, 1983). To show that these c roots are again distinct, we create a contradiction beginning from the assumption that $z_i \neq 0$ is a repeater. Then the derivative of the left-hand side of (4) should vanish at z_i. The derivative is easily seen to be

$$(z_i^{c-1} exp[\lambda(1 - z_i)])(\lambda z_i - c),$$

which can only be 0 if $z_i = c/\lambda$. But this cannot happen since c/λ is the reciprocal of the system utilization, which must be less than 1 for ergodicity, and therefore z_i is outside the unit circle. So the roots inside and on the unit circle are simple again.

To find the roots, the following technique was originally developed by Downton (1955), expanded by Powell (1985), and later improved systematically by Chaudhry together with various collaborators in 1986 and 1987. Equation 5 is clearly equivalent for all $n = 1, 2, ..., c$ to

$$z = exp[-\lambda(1 - z)/c] exp(2\pi ni/c), \tag{6}$$

where i is the squareroot of -1 and $exp(2\pi ni/c), n = 1, 2, ...c$, are the c complex roots of unity. Chaudhry, Madill and Brière (1987) have shown that, for each n, (6) has exactly one root inside the unit circle. Powell (1985) used Newton's method in his work to find that root; but Chaudhry et al. have had more (and complete) success using Muller's method, never having encountered any numerical difficulties. We present more on a similar problem later.

Further results are available when there is random, bulk input, i.e., when the model is $M^{(X)}/D/c$. Equation 5 must now incorporate information on the probabilistic nature of the batch-input sizes. This is done through their probability generating function, say $X(z)$. Then the characteristic equation becomes

$$z^c = exp(-\lambda[1 - X(z)]),$$

with c roots found to be inside and on the unit circle.

2.4 $G/M/1$ and $G/E_K/1$ or $G^{(K)}/M/1$

The waiting-time distribution function for the general single-server problem with exponential service requires the lone real root on (0,1) for the (fundamental branching process) equation

$$z = \sum_{i=0}^{\infty} b_i z^i = A^*[\mu(1 - z)] = \beta(z)$$

where A^* is the Laplace-Stieltjes transform (LST) of the interarrival times and β is the probability generating function (pgf) of the arrivals during service (see Gross and Harris, 1985). The LST is easily shown to be monotone nondecreasing and convex, and thus the root is readily obtainable. For example, the well-known Newton-Raphson approximation method is guaranteed to converge because of the convexity.

The problem becomes more interesting when Erlang(K) service times are used instead. Here the roots need to be generally located and then found for

$$z^K = \sum_{i=0}^{\infty} b_i z = A^*[\mu(1 - z)] = \beta(z) \tag{7}$$

or, using $z = rexp(i\theta)$ and principal values for all log calculations,

$$Kln(r) + iK\theta = ln\{A^*[\mu(1 - r\,exp(i\theta))]\} + 2\pi ni \tag{8}$$

for $n = 1, 2, \ldots, K$ (see Chaudhry, Madill and Brière, 1987). There is clearly a root at unity, and by Rouché's theorem, we can once again show that there are K others inside the unit circle $\mid z \mid = 1$. For each n now, Chaudhry, Jain and Templeton (1987) note that there is a unique root with absolute value less than 1, using the complex version of the monotonicity and convexity argument employed for the $G/M/1$.

Note that the characteristic equation for the model $G^{(X)}/M/1$ bears a great similarity to (7). We know that z^K is the probability generating function of the batch-input-size distribution since the only possible size is identically K. Rewrite (7) then as

$$1 = \beta(z)(1/z)^K.$$

Then replace the term $(1/z)^K$ by the input-size pgf $X(z)$ evaluated at $1/z$, thus giving the c.e.

$$1 = \beta(z)X(1/z)$$

The roots of (8) are found by separately solving its real and imaginary portions. We know that there is a unique answer when (8) is evaluated for individual values of n. But prior to this work, it was assumed that roots obtained in this manner could be the same. We have, however, now been able to show that repetition is not possible! We provide our proof of this assertion in Section 3.2.

Partial results on the uniqueness of all the roots for this model type are available. It is, in fact, known that the roots are totally unique when the interarrival times are Erlang(J) distributed (see Chaudhry and Templeton, 1983). In this case, Equation 7 becomes

$$z^K = [J\rho/(J\rho + K(1 - z))]^J. \tag{9}$$

Again, we need to have the derivative vanish at any repeated root, say z_*. This is equivalent to requiring that

$$z_*^{K-1} = J(J\rho)^J[J\rho + K(1 - z_*)]^{-(J+1)}. \tag{10}$$

When (9) is divided by (10), we see that

$$z_* = (J\rho + K)/(J + K).$$

When this value is substituted back into Equation 9, we find that the condition for repetition is equivalent to requiring that

$$\rho^c = \rho c + (1 - c). \tag{11}$$

The right-hand side of (11) is a straight line in ρ, with y-intercept of 1-c and slope of c, while the left-hand side is a simple monomial with integral power c. The two functions intersect only at $\rho = 1$, which would violate the condition for ergodicity, so that the assumption of repetition must be false.

But not much more has been generally known about the effect of the form of the interarrival distribution? Chaudhry, Jain and Templeton (1987) tried the method of Equation 8 on a large number of cases and did not encounter any problems. They used the *QPACK* software package developed by Chaudhry and his collaborators (Chaudhry, 1988). However, such computational testing clearly cannot be exhaustive. We address this question in a more complete fashion in Section 3.

The waiting-time distribution function for the $G/E_K/1$ system (see, for example, Chaudhry and Templeton, 1983) is

$$W_q(t) = 1 - C \sum_{i=1}^{K} a_i r_i exp[-\mu(1 - r_i)t]/(1 - r_i)$$

where C is the arrival-point probability that there are no phases present, and the $\{r_i\}$ are the K complex roots inside the unit circle. The mixing constants $\{a_i, i = 1, 2, ..., r\}$ are found by the formula

$$a_i = \prod_{j \neq i} r_i/(r_i - r_j) \qquad (r_i \neq r_j).$$

2.5 $M/E_K^{(Y)}/1$, $M/G^{(Y)}/1$ and $E_K/G/1$

For the more general batch-service system with Poisson input, Erlang(K) distributed service times and random batch sizes, the characteristic equation whose roots we require is

$$\frac{\mu^K}{(\mu + \lambda - \lambda z)^K} Y(1/z) - 1 = 0, \tag{12}$$

where Y(z) is the probability generating function for the random service batch (defined to have finite support). If we write Y(z) as the polynomial

$$Y(z) = y_1 z + y_2 z^2 + y_3 z^3 + \ldots + y_b z^b,$$

then Equation 12 can be rewritten as

$$z^b = [1 + \frac{\lambda}{\mu}(1-z)]^{-K}(y_1 z^{b-1} + y_2 z^{b-2} + \ldots + y_b).$$

Clearly, 1 is a root; in addition, K roots are outside the unit circle, while b-1 are inside and on. All of the roots inside are again distinct and easy to obtain (see, for example, Chaudhry and Templeton, 1983). But, as noted in our $M/M^{(K)}/1$ discussion, it is the roots outside which are critical. The model $M/G^{(Y)}/1$ is the generalization of the E_K service model and has any number of roots outside the unit circle (e.g., for G=M, there is 1).

Brière and Chaudhry (1989) have analyzed in detail relatives of this model where service is instead either hyperexponential, uniform or constant. Results are once more quite favorable.

To show the further equivalence of this model type to the $E_K/G/1$, let $Y = K$ in the $M/G^{(Y)}/1$ and then convert the constant batch size and Poisson input over to an Erlang(K) input. The resultant characteristic equation is thus

$$z^K = B^*[\lambda(1-z)], \tag{13}$$

where B* is the Laplace-Stieltjes transform of the service-time distribution. Note the clear analogy to the c.e. for $G/E_K/1$ given by Equation 7. Equation 13 (with K=b) is also the characteristic equation of the model $M/G^{a,b}/1$, where the server takes batches of size b if available and otherwise waits until at least a customers are waiting.

It is well known that (13) has K roots inside and on the unit circle, including the root z=1 (see, for example, Abolnikov and Dukhovny, 1987). We show here that z=1 is simple using the usual derivative test. To do so, we evaluate

$$Kz^{K-1} = -\lambda dB^*(\lambda - \lambda z)/dz$$

at z=1 and find that

$$K = -\lambda(-1/\mu) = K\rho,$$

or $\rho = 1$, which is a contradiction. Hence z=1 cannot be a double root.

When the service times are deterministic, we can further show that all K-1 roots inside or on the unit circle are, in fact, strictly within. This follows when we rewrite (13) as

$$z^K = e^{K\rho(z-1)}$$

and assume that z_i has absolute value of 1 but is not precisely equal to 1. Then we see that

$$1 = e^{K\rho(z_i-1)},$$

which implies that $Re(z_i) = 0$ and thus that $z_i = 1$. But this is contrary to our earlier verification that the root $z = 1$ is simple.

3 FURTHER RESULTS AND COMPUTATIONAL EXPERIENCES

In this section, we provide what we feel is (fairly) complete evidence on the ultimate efficacy of rootfinding in queueing. For each of the models of Section 2, we shall expand on the theory of root location and repetition, and then computationally push each problem to its extremes to reinforce the notion that roots can always be found for queueing models.

3.1 $M^{(K)}/M/1$

First, we report on further computational experience with the Markovian batch-arrival model. (We do not do likewise for the Markovian batch-service queue since it requires the location of only one root and that is not difficult.) We are thus working with the polynomial defined in Equation 3, and have used the software developed by Chaudhry and Hasham (1987). The key parameters are thus the bulk size K and the traffic intensity. We have selected the extreme values of the traffic intensity for our experiments to be .05, .1, .9 and .95. We have also run at the value .5 as a calibration. The values of K chosen were 10, 25, 50 and 100. The results are presented below in Table 1 for runs performed on an 80286-based, 12 mHz AT clone with a math coprocessor. Here, as for subsequent examples, root values have been verified to be correct using the IMSL package ZANLY for analytic function rootfinding.

Table 1

$M^{(K)}/M/1$ Computations

Intensity	*Batch Size*	*Run Time (min:sec) for Roots*
.50	10	0:16
	25	0:21
	50	0:28
	100	0:40
.05	10	2:03
	25	2:15
	50	2:25
	100	2:41
.10	10	1:04
	25	1:12
	50	1:19
	100	1:34
.90	10	0:12
	25	0:15
	50	0:22
	100	0:35
.95	10	0:11
	25	0:15
	50	0:22
	100	0:34
	500	15:41

3.2 $G/E_K/1$

We know that this model has K roots inside the unit circle, and as promised, we now show that these values are indeed unique.

Theorem 1 *The roots of the characteristic equation of the $G/E_K/1$ model (or, equivalently, the $G^{(K)}/M/1$) are unique, with one real for K odd and two real for K even.*

Proof: Use (7) in the form $z^K = \beta(z)$. Then by a geometric argument essentially the same as that for the $G/M/1$ used in Figure 5.1 of Gross and Harris (1985), it follows that there exists a unique *real* root in (0,1) for *all* K when

$$\beta'(1) > [d(z^K)/dz]_{z=1}.$$

But this is equivalent to

$$\frac{\mu}{\lambda} > K \qquad \text{or} \qquad \frac{K\lambda}{\mu} < 1,$$

which is true from ergodicity. [For K even, it is easy to see that there is an additional real root in (-1,0).]

Next, remember from Section 2.4 that Equation 8 has a unique root inside the unit circle for each $n = 1, ..., K$; call it (r_n, θ_n). But it is also true that (8) has a unique (possibly non-integer) value, n_i, for each pair (r_i, θ_i). Thus if we assume for $i \neq j$ that $(r_i, \theta_i) = (r_j, \theta_j)$, it follows that $n_i = n_j$. But *this contradicts the uniqueness of* (r, θ) *for each* n. *Therefore* all K *pairs* of roots (r_n, θ_n) must be different. □□.

Because of this uniqueness, we see that the waiting-time distribution function of Section 2.4 for the line delays is a generalized hyperexponential. The mixing constants $\{a_i\}$ of the CDF formulation are guaranteed to be distinct and easily computed.

To show the ease with which $G/E_K/1$ roots may be found, we consider examples using the most general distribution classes found in queueing, namely, the phase types of Neuts (see Neuts, 1981), the generalized hyperexponentials of Harris (see Botta, Harris and Marchal, 1987) and the Coxian distributions (Cox, 1955). If the results for the most difficult problems using these kinds of distributions (with each class known to be dense in the set of all CDFs) are favorable, then we become much more comfortable in rootfinding efforts using any other interarrival distributions. Two examples of each class have been studied, and the results are presented in the following. The first PH example is taken from Botta, Harris and Marchal and has CDF

$$F(t) = 1 - 1.293e^{-4.846t} + 0.343e^{-4.195t} - 0.05e^{-0.959t}.$$

Equation 7 now becomes

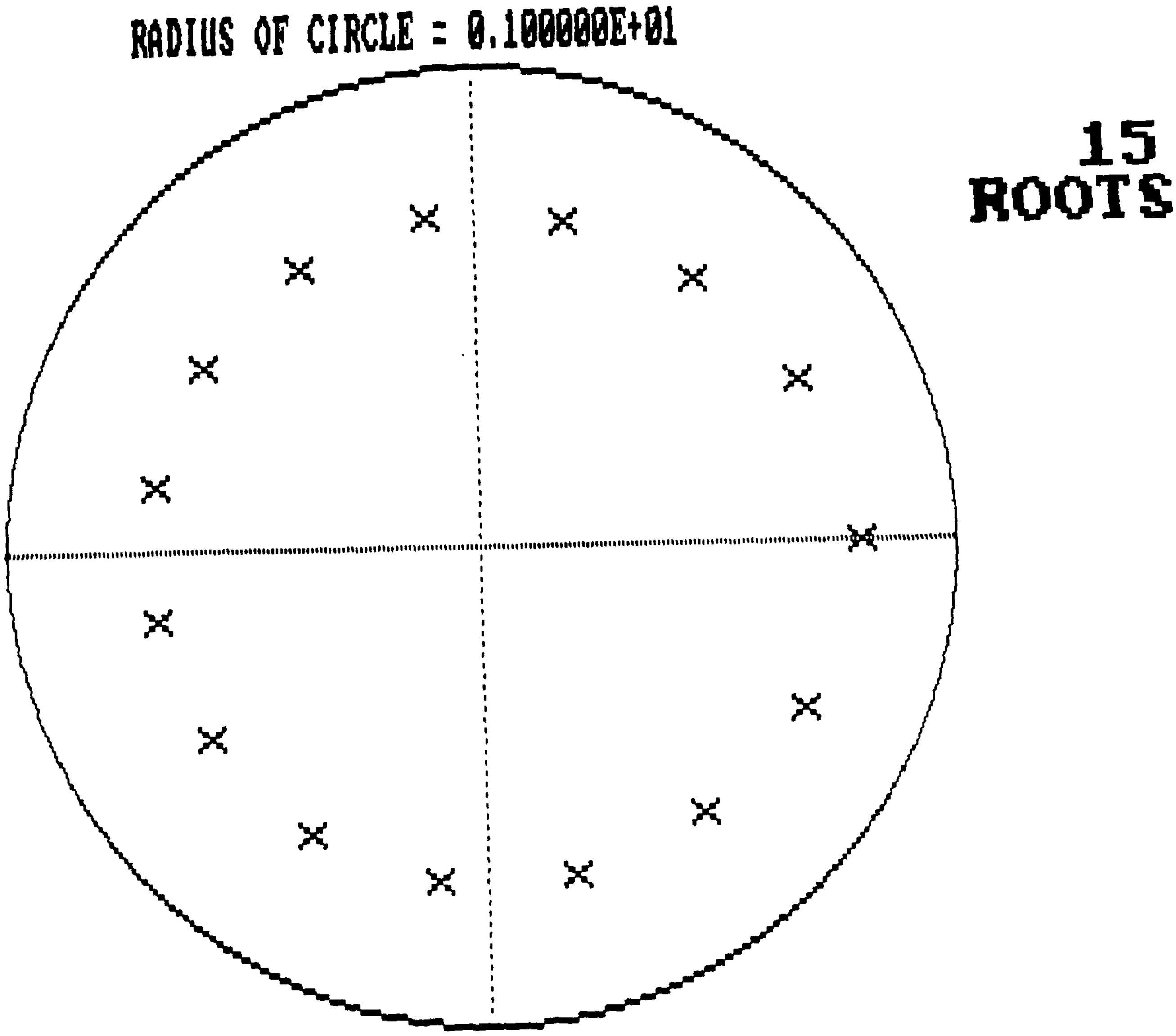
$$z^K = 6.266/[4.846+\mu(1-z)]-1.439/[4.195+\mu(1-z)]+.04795/[.959+\mu(1-z)].$$

We have chosen three medium to large values of K, namely, 10, 15 and 30, remembering that it is K that determines the number of roots we need to find. Each K has been paired up with a high and low traffic intensity. Recall that we are to determine K roots inside the unit circle. We have used the Chaudhry *QPACK* software, with all of the roots found quickly in each case. Our experiences are given in Table 2, and a plot is presented as Figure 1 of the actual location of the roots over the unit circle for one of the $K = 15$ examples ($\rho = .1$).

Table 2
$PH/E_K/1$ Computations

Batch Size	μ	*Intensity*	*Run Time (min:sec) for Roots*
10	46.0	.91522	0:21
10	421.0	.10000	0:21
15	70.0	.90214	0:30
15	631.5	.10000	0:28
30	140.0	.90214	0:49
30	1263.0	.10000	0:49

Figure 1: Root Location, I



The first GH illustration is an example offered by Botta, Harris and Marchal, which turns out to be also PH, though not a mixed generalized Erlang. This is

$$F(t) = 1 - 6e^{-4t} + 13e^{-3t} - 8e^{-2t},$$

with $E[T] = 7/6$. The resultant version of (7) which is found when the appropriate transforms are taken is

$$z^K = 24/[4 + \mu(1 - z)] - 39/[3 + \mu(1 - z)] + 16/[2 + \mu(1 - z)].$$

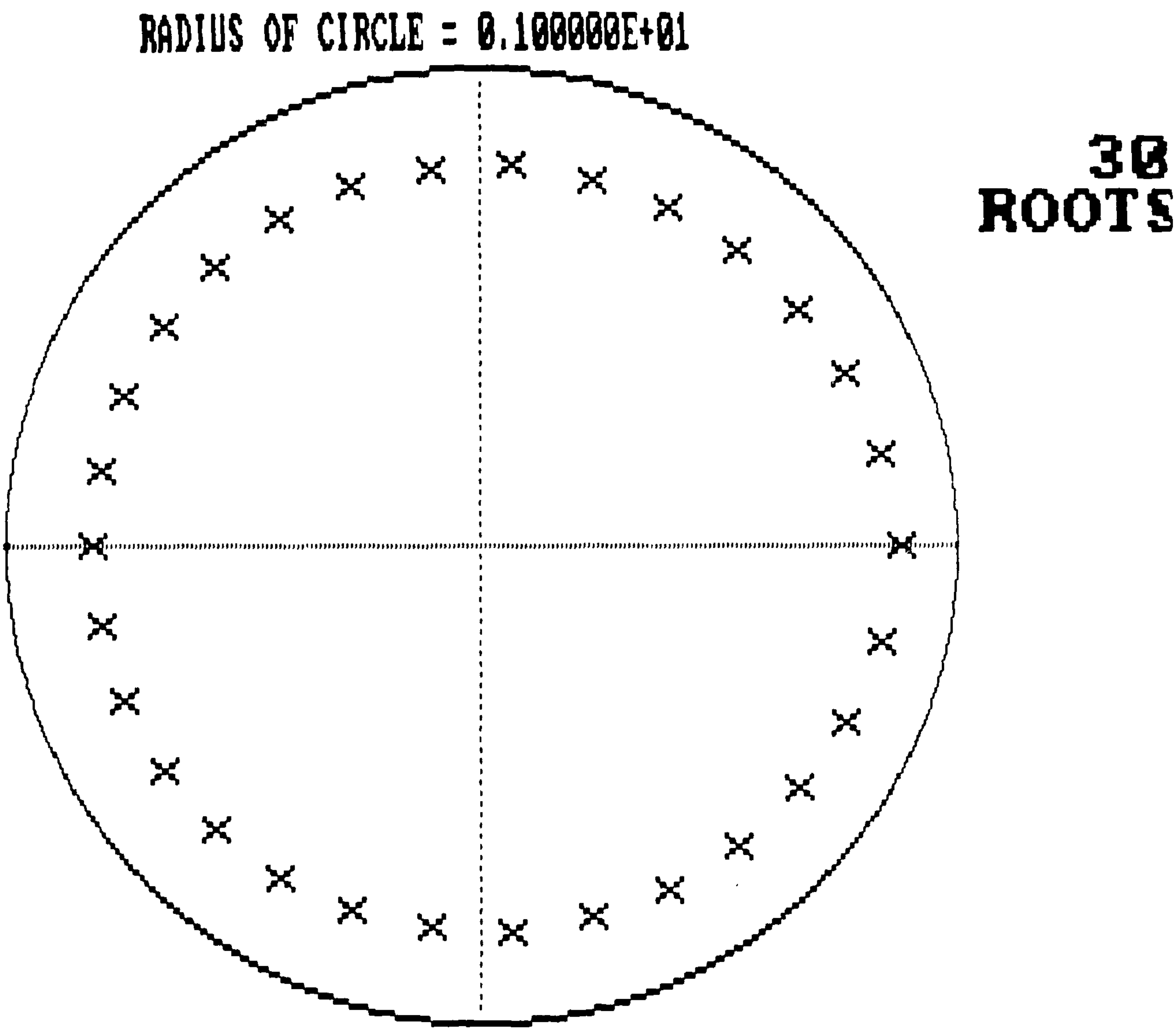
The problem is again most challenging when K is at least fairly large. So once more we have set K in separate runs to be 10, 15 and 30 and used both a high and low traffic intensity for each value of K. We have again used the Chaudhry *QPACK* software and all roots were found quickly in each case, with the results displayed in Table 3. For illustration, a plot of the actual location of the roots over the unit circle for the final of these six examples (with $K = 30$ and $\rho = .10006$) has been included as Figure 2.

Table 3

$GH/E_K/1$ Computations

Batch Size	μ	*Intensity*	*Run Time (min:sec) for Roots*
10	9.00	.95238	0:21
10	85.00	.10084	0:21
15	14.01	.91771	0:52
15	128.00	.10045	0:29
30	29.00	.88670	1:30
30	257.00	.10006	0:48

Figure 2: Root Location, 1]



We also experimented with a second GH distribution taken from Botta, Harris and Marchal:

$$F(t) = 1 - 4e^{-t} + 6e^{-2t} - 3e^{-3t}.$$

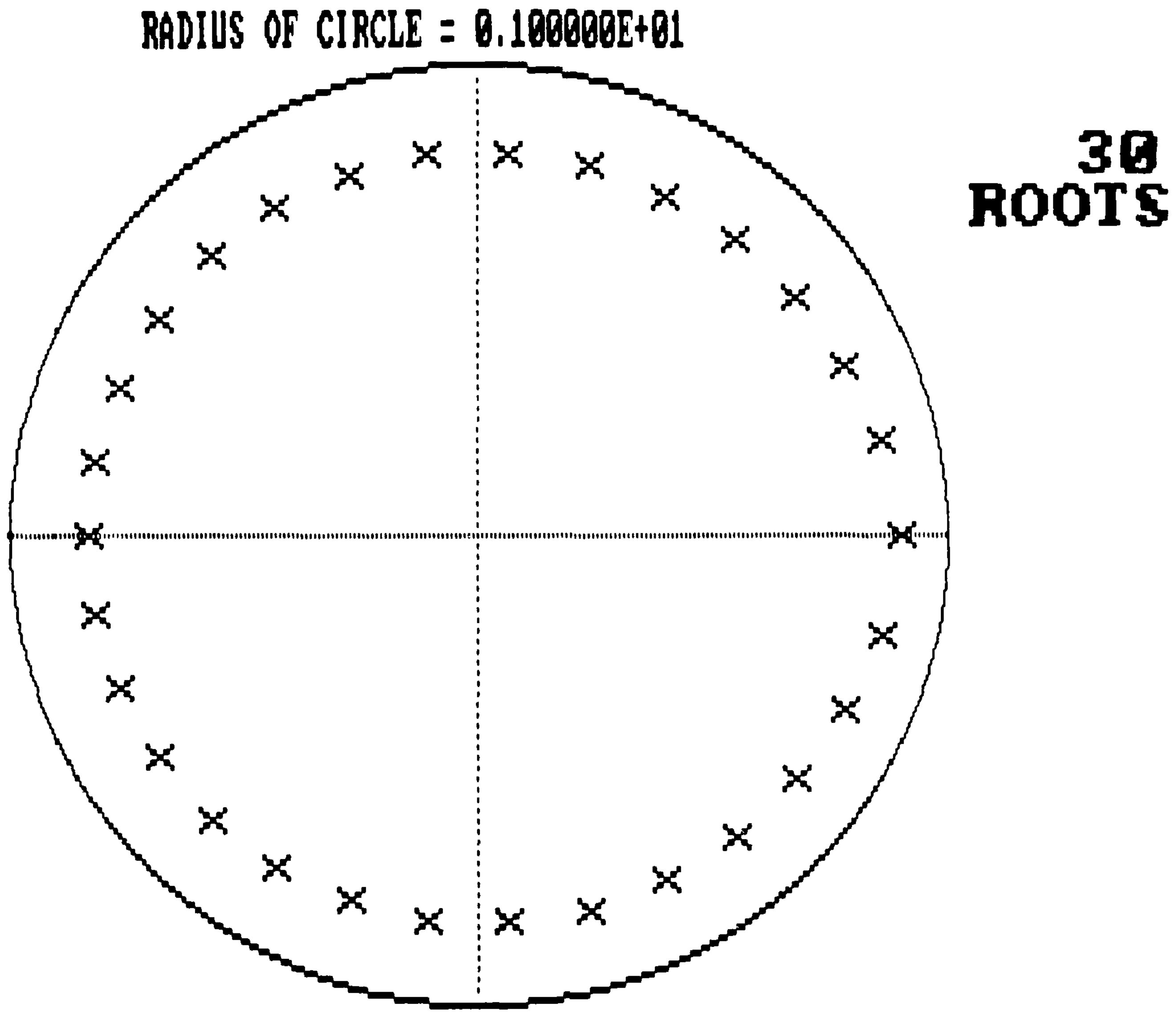
This particular one is of potential special interest since it turns out that it is not also a phase-type CDF. The mean interarrival time $1/\lambda$ is found to be 2. We have again chosen a range of medium to large values of K, namely, 10, 15 and 30, and have used a pair of high and low traffic intensities for each. These results are displayed in Table 4, with a plot of the $K = 30, \rho = .1$ case presented as Figure 3.

Table 4

Second $GH/E_K/1$ Example

Batch Size	μ	*Intensity*	*Run Time (min:sec) for Roots*
10	5.60	.89286	0:21
10	50.00	.10000	0:21
15	8.15	.92025	0:53
15	75.00	.10000	0:28
30	16.66	.90004	1:31
30	150.00	.10000	0:49

Figure 3: Root Location, III



Next, we went back to a phase-type input process and used the harmonic distribution

$$F(t) = 1 + e^{-2.8846}[.3868sin(.5897t) + .1729cos(.5897t)]$$

$$-1.1729e^{-.2307t}$$

as an illustration. Results were again excellent, with all roots found quickly and efficiently - see Table 5.

Table 5
Second Phase-Type Example

Batch Size	μ	*Intensity*	*Run Time (min:sec) for Roots*
10	20.000	.10000	0:25
10	2.200	.90909	0:26
15	30.000	.10000	0:35
15	3.330	.90090	0:38
30	60.000	.10000	1:03
30	6.667	.89996	1:06

The final two examples of this section are Coxian distributions, with the requisite rational LSTs. Each of these is also neither a GH or PH distribution. The presence of an harmonic term guarantees that the density cannot be GH, while the fact that the df hits the t-axis at at least one point guarantees that we do not have a phase-type problem as well. The first case here is the density

$$f(t) = 10e^{-2t}/[1 \quad cos(t)].$$

All of the roots were once again found quickly and efficiently - see Table 6.

Table 6

Second $PH/E_K/1$ Example

Batch Size	μ	*Intensity*	*Run Time (min:sec) for Roots*
10	50	.28571	0:24
10	18	.79365	0:24
15	189	.11337	0:32
15	23	.93168	0:59
30	371	.11552	1:35
[illegible]	48	.89286	1:23

The concluding example is a Coxian distribution function with an atom at the origin:

$$F(t) = 1 - \frac{1}{3}(e^{-t} + e^{-2t}).$$

The results follow.

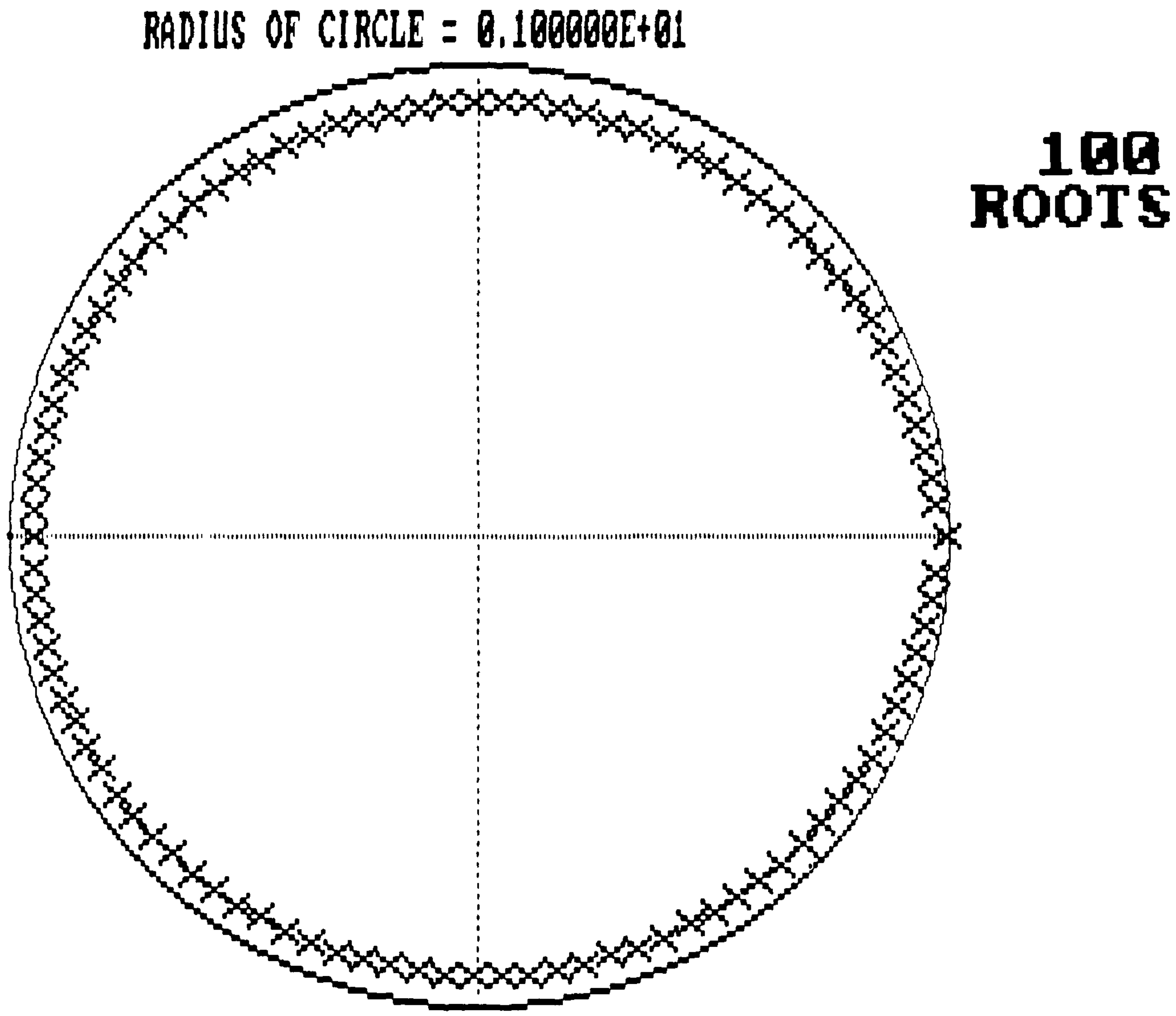
Table 7

Second Coxian Example

Batch Size	μ	*Intensity*	*Run Time (min:sec) for Roots*
10	200	.10000	0:19
10	22	.90909	0:20
15	300	.10000	0:26
15	32	.93750	0:27
30	600	.10000	0:46
30	62	.96774	0:45

As a last test case for this example, we used a batch size of 100 and service rate of 220 (giving a traffic intensity of .90909). The run took 2 minutes and 11 seconds, and the roots are displayed in Figure 4.

Figure 4: Root Location, IV



3.3 $G/GE_K/1$

An obvious extension of the $G/E_K/1$ model is the broader class of models where service times are generalized Erlang(K), that is, they are found as the convolution of K independent but not identically distributed exponential random variables. For this very large and dense class, we can show as in the prior model that there is exactly one real root of the characteristic equation for K odd and two real roots for K even. Unfortunately, uniqueness may not obtain here, but root location is not difficult anyway. We thus present the following result on real roots as Theorem 2.

Theorem 2 *The characteristic equation of the $G/GE_K/1$ model has a unique positive real root when K is odd and two unique real roots when K is even.*

Proof: The c.e. for this model is more commonly known from the $G/G/1$ formulation as

$$A^*(-s)B^*(s) = 1, \tag{14}$$

where A^* and B^* are the Laplace-Stieltjes transforms of the interarrival and service-time distributions, respectively. From Gross and Harris (1985), for example, we know that (14) has K roots with negative real parts. Rewrite (14) as

$$A^*(s) = \frac{1}{B^*(-s)} = \prod_{i=1}^{K} \frac{\mu_i - s}{\mu_i}, \tag{15}$$

where the generalized Erlang's phase rates have been placed in ascending order as $\mu_1, \mu_2, \ldots, \mu_K$. Then let $s = \mu_K(1 - z)$ and substitute into (15). It follows that

$$A^*[\mu_K(1 - z)] = \prod_{i=1}^{K} \frac{\mu_i - \mu_K + \mu_K z}{\mu_i}.$$

But we have thus effectively reduced the problem to one which is nearly identical to that of the $G/E_K/1$ model of Equation 7, namely, an LST evaluated at $\mu(1 - z)$ equal to a polynomial of degree K. The geometric argument of Theorem 1 now goes through, with appropriate real solutions as long as the expected number of arrivals per service time is less than 1. □□ .

3.4 $E_K/G/1$ and $M/G^{a,b}/1$

Remember from (13) that these models have characteristic equation

$$z^b = z^K = B^*(\lambda - \lambda z),$$

where B* is the Laplace-Stieltjes transform of the service-time distribution.

We next offer a numerical example for this model type. Consider an $E_K/G/1$ problem in which service is generalized Erlang. Let the GE distribution have phases with mean rates $\mu_1 = 12$, $\mu_2 = 6$ and $\mu_3 = 4$. Then the equation of interest becomes

$$z^K = \frac{1}{1+12\lambda(1-z)}\frac{1}{1+6\lambda(1-z)}\frac{1}{1+4\lambda(1-z)}.$$

We have varied K in our usual way, combined with a variety of input rates and thus traffic intensities. Roots have once more been found quite easily. The results are offered in Table 8.

Table 8

$E_K/G/1$ Example

Batch Size	λ	*Intensity*	*Run Time (min:sec) for Roots*
10	2	.1	0:23
10	18	.9	0:22
15	3	.1	0:31
15	27	.9	0:31
30	6	.1	0:51
30	54	.9	0:50

4 CLOSING REMARKS

We have tried to show that rootfinding in classical queueing models is not a difficult matter. Our view of this is based on the facts that the critical roots typically appear singly in easily located regions and that our empirical numerical experience has always been most favorable. While it is true that we have not examined multi-server queues in great detail, rootfinding in those cases is generally quite similar to the single-channel models with similar input and service-time-distribution combinations. There are clearly problems whose basic characteristics are so involved that rootfinding is almost guaranteed to be perverse. But we believe that the models we have explored are very comprehensive in their application and that lessons from their solution are very useful in the effective solution of the more complicated models.

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